TECHNICAL STABILITY OF THE DYNAMIC STATES OF AN EXTENDED ROD WITH A VARIABLE CROSS SECTION, MOVING LONGITUDINALLY IN A FLUID

K. S. Matviichuk

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The stability of the motion of very extended dynamic systems can in many cases be studied as a problem in the stability of long rods. Long rods and rod structures interacting with an internal or external stream of fluid are used extensively in practice. Under the action of an external force such rods may be susceptible to considerable displacements. It is thus desirable to use the necessary nonlinear relations when studying the dynamic behavior of such systems [1, 2]. When the forces due to the interaction of the moving rod with the external stream of fluid are taken into account the problems are more involved than the traditional problems considered in the mechanics of rods.

We have studied the conditions for technical stability [3-8] of a long rectilinear rod with a variable cross section during its longitudinal transportation in a moving ideal fluid. The process is described by a nonlinear system of three different equations in partial derivatives with inhomogeneous boundary conditions. Sufficient conditions were obtained for the technical stability of a system in finite and infinite intervals of time and with an asymptotic technical stability. Conditions under which a system may lose stability are indicated. A formula is found for the critical velocity of a rod in a fluid. The results were obtained by comparison involving the Lyapunov method [4, 6-11].

1. Formulation of the Problem. We consider a long flexible rod AB with a variable cross section, whose axis is rectilinear in the initial state. Suppose that this rod is transported longitudinally in an ideal incompressible fluid for a given time $I_1 = [t_0, K] \subset I \equiv [t_0, +\infty)$ ($t_0 \geq 0$, $K = \text{const} > 0$) along a horizontal rectilinear trajectory with a given velocity $\mathbf{v}$. We consider the instantaneous configuration of the rod [2]. The rod is assumed to be a homogeneous isotropic solid, which is subject to nonlinear geometric deformation with small strains. We examine the case when a fluid flows past the rod, positioned asymmetrically relative to the stream of fluid. The resultant hydrodynamic force $\mathbf{F}$ then has the same direction as the stream. The force has two components: $\mathbf{F}_c = (F_{1c}, F_{2c}, F_{3c})$ is the hydrodynamic drag force directed along the stream and $\mathbf{F}_p = (F_{1p}, F_{2p}, F_{3p})$ is the lift force directed perpendicular to the stream. We introduce the following notation for the rod: $m(s)$ is mass per unit length as a function of s; ρ is the density of the material; $S(s)$ is the area of any cross section as a function of s; ℓ is the length; h is the average thickness; $\mathbf{q} = (q_1, q_2, q_3)$ is the vector of the applied external distributed forces: $\mathbf{u}(t, s) = \{u_1(t, s), u_2(t, s), u_3(t, s)\}$ is the dimensional displacement vector of any point on the axial line; $\varepsilon = s\ell$ is the dimensional scalar coordinate of any point of the undeformed axial line: $s \in D \equiv [0, 1]$; t is a dimensionless time variable; τ is the dimensional time; ν is Poisson's ratio; E is Young's modulus; P_m is the force of the weight per unit length; P_A is the Archimedean buoyancy; and Ω is the thrust of the propulsive device. The leading end A of the rod has a swivel fastening with a propulsive device while a body Π acts on the other end B in an equalizing manner relative to the horizontal; for this body we introduce the notation $\mathbf{q}_\Pi$, ρ_f, $\tilde{h}$, V_Π, $\mathcal{F}_A = g\rho_f V_\Pi$, and ξ_c, which are its weight, density, a characteristic linear parameter, volume, Archimedean buoyancy, and the distance between the center of mass of the body Π and the point B; g is the free fall acceleration. By $\mathbf{e}_{10}$, $\mathbf{e}_{20}$, and $\mathbf{e}_{30}$ we denote an orthogonal local system of unit vectors for the rod in the unperturbed state. The vector $\mathbf{e}_{10}$ is directed along the axial line of the rod and toward the transportation. Suppose that $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the vectors of the orthogonal local system of coordinates in the instantaneous configuration of the rod; $\mathbf{e}_2$, and $\mathbf{e}_3$ are direct along the principal axis of its cross section. The systems $\mathbf{e}_{i0}$ and $\mathbf{e}_i$ ($i = 1, 2, 3$) both have a right-handed orientation. In the

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instantaneous configuration of AB we distinguish an arbitrary element of the rod $d\varepsilon$, which is bounded by the cross sections ε, $\varepsilon + d\varepsilon$. The origins O_1 and O_1^* of the systems $\mathbf{e}_{i0}$ (i = 1, 2, 3) and $\mathbf{e}_i$ (i = 1, 2, 3) in the unperturbed state of the rod coincide and are assumed to lie at the midpoint of the axial line of the element $d\varepsilon$. The radius-vector of O_1^* at any time τ is $\mathbf{r}(\tau, \varepsilon) = \mathbf{r}(\varepsilon) + \mathbf{w}(\tau, \varepsilon)$. Here $\mathbf{r}(\varepsilon)$ is the radius-vector of O_1 and $\mathbf{w}(\tau, \varepsilon)$ is the displacement vector of O_1 for any arbitrary position of O_1^* in the instantaneous configuration. For any point of the axial line we have $d\mathbf{r}/d\tau = \partial\mathbf{w}/\partial\tau$, $d^2\mathbf{r}/d\tau^2 = \partial^2\mathbf{w}/\partial\tau^2$. The element $d\varepsilon$ is acted on by an inertial force

$$d\mathbf{J}_{\text{in}} = -m(\varepsilon)\frac{\partial^2\mathbf{w}}{\partial\tau^2}\,d\varepsilon,$$

where the fact that $\mathbf{v}$ is a constant has been taken into account. In the general case an element of rod can be acted upon by a distributed force $\mathbf{q}$ and a moment $\mu_0 = (\mu_{10}, \mu_{20}, \mu_{30})$, which are not associated with the stream, as well as by the hydrodynamic force $\mathbf{F}$ and the moment $\boldsymbol{\mu}_a$, which arise when the rod interacts with the external stream of fluid [2, 12, 13]. When the rod moves the hydrodynamic forces acting on it depend on the square of the relative velocity $\mathbf{v}_{0T}$ of the stream [12, 13]: $\mathbf{v}_{0T} = \mathbf{v}_0 - \mathbf{v}_1$, $\mathbf{v}_1 = \mathbf{v} + \partial\mathbf{w}/\partial\tau$ is the velocity vector of the points on the axial line of the rod and $\mathbf{v}_0$ is the vector of the absoltue velocity of the stream. The hydrodynamic force can be represented as the sum of two forces: $\mathbf{F}_c = \mathbf{q}_n + \mathbf{q}_1$ ($\mathbf{q}_1$ is directed along a tangent to the axial line of the rod, i.e., along the direction of $\mathbf{e}_1$ and $\mathbf{q}_n$ is directed along the normal to the axial line, i.e., perpendicular to $\mathbf{q}_1$). In the general case the element of the rod has an angular velocity ω and is acted on by the moment of inertia

$$d\mathbf{M}_{\text{in}} = -\frac{\partial}{\partial\tau}(\mathbf{J}\boldsymbol{\omega})\,d\varepsilon.$$

We assume that ε remains constant during the motion [12]. We write matrix $\mathbf{J}$ as

$$\mathbf{J} = \rho\mathbf{J}_0,\ \mathbf{J}_0 = \begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix},$$

where J_i (i = 1, 2, 3) are the moments of inertia of the cross section with respect to the principal axis of the cross section (i = 2, i = 3) and the axial line of the rod (i = 1). Suppose that the flexural center of the rod coincides with its center of gravity. We use the d'Alembert principle, whereupon we obtain

$$m(\varepsilon)\frac{\partial\mathbf{v}}{\partial\tau} = \frac{\partial\mathbf{Q}}{\partial\varepsilon} + \mathbf{R}; \tag{1.1}$$

$$\frac{\partial}{\partial\tau}(\mathbf{J}\boldsymbol{\omega}) = \frac{\partial\mathbf{M}}{\partial\varepsilon} + \mathbf{e}_1\times\mathbf{Q} + \boldsymbol{\mu},\quad \boldsymbol{\mu} = \boldsymbol{\mu}_0 + \boldsymbol{\mu}_a. \tag{1.2}$$

Here $\mathbf{R} = \mathbf{R}(\mathbf{v}, \tau, \varepsilon)$ is the principal vector of all the external forces acing on the rod; $\mathbf{Q}$ is the vector of internal forces of the element of rod: $\mathbf{Q} = Q_1\mathbf{e}_1 + Q_2\mathbf{e}_2 + Q_3\mathbf{e}_3$; Q_1 is the axial force and Q_2 and Q_3 are intersecting forces. The vector of the internal moments $\mathbf{M} = M_1\mathbf{e}_1 + M_2\mathbf{e}_2 + M_3\mathbf{e}_3$ (M_1 is a torsional moment and M_2 and M_3 are bending moments). The vectors $\mathbf{Q}$ and $\mathbf{M}$ are statically equivalent to the respective stress vectors [2]. For rods of variable cross section $m(s) = m_0(0)n_0(s)$ [$n_0(s)$ is a dimensionless function, $m_0(0) = \rho S_0$, and S_0 is the area of a fixed cross section]. The area of any cross section of the rod is $S(s) = S_0 \cdot n_0(s)$. Using the general equations (1.1) and (1.2), we write the equations of motion for the considered case in the projections onto the unit vectors $\mathbf{e}_{i0}$ (i = 1, 2, 3). Corresponding to an arbitrary point Z outside the longitudinal axis, belonging to the cross section through the point O_1 of the element $d\varepsilon$, in the instantaneous configuration is the vector $\mathbf{U} = \mathbf{U}(\varepsilon, \eta, \zeta)$ of its displacement to the position Z^* in the case of cubic strain. The radius-vector of Z is $\mathbf{R}_0 = \mathbf{r}(\varepsilon) + \eta\mathbf{e}_{20} + \zeta\mathbf{e}_{30}$, and that of Z^* is $\mathbf{R}^* = \mathbf{R} + \mathbf{U} \equiv \mathbf{r}(\varepsilon) + \eta\mathbf{e}_{20} + \zeta\mathbf{e}_{30} + \mathbf{U}$. The square of the infinitesimal distance between the two points in the initial configuration of the rod is $dl^2 = d\varepsilon^2 + d\eta^2 + d\zeta^2 = d\mathbf{R}_0 d\mathbf{R}_0$ and that of the infinitesimal distance in the instantaneous configuration is

$$(dl^*)^2 = d\mathbf{R}^* d\mathbf{R}^*,\ d\mathbf{R}^* = \frac{\partial\mathbf{R}^*}{\partial\varepsilon}d\varepsilon + \frac{\partial\mathbf{R}^*}{\partial\eta}d\eta + \frac{\partial\mathbf{R}^*}{\partial\zeta}d\zeta.$$

Writing the difference $(d\ell^*)^2 - d\ell^2$ in terms of $\mathbf{U}$ on the one hand and in terms of the strain tensor ε_{ij} (i, j = 1, 2, 3) on the other hand, we find the representation of the components ε_{ij} by the relations of the derivatives $\partial\mathbf{U}/\partial\varepsilon$, $\partial\mathbf{U}/\partial\eta$, $\partial\mathbf{U}/\partial\zeta$. The tensor of the stresses σ_{ij} (i, j = 1, 2, 3) at the point Z^* is given by the known relations of [2] in accordance with Hooke's law. The displacement vector is chosen in the form [2]

$$U_1 = w_1 + \widehat{a}_2\eta + \widehat{a}_3\zeta,$$
$$U_2 = w_2 + \widehat{b}_2\eta + \widehat{b}_3\zeta,\ U_3 = w_3 - \widehat{b}_3\eta + \widehat{b}_2\zeta, \tag{1.3}$$

where $\hat{a}_2$, $\hat{a}_3$, $\hat{b}_2$, $\hat{b}_3$ are real coefficients, which characterize the small angles of rotation that generally depend on the variable ε. The notation of $\mathbf{U}$ in the form (1.3) is consistent with the hypothesis of plane sections, namely: sections that are perpendicular to the axis of the rod prior to deformation remain plane, but no loner necessarily orthogonal to the axis of the rod. Indeed, with the approximation (1.3) we obtain an affine transformation of points lying in the plane of the section perpendicular to the axis of the rod before deformation; as a result of the transformation in the instantaneous configuration these points are once again in one plane and straight-line segments are correspondingly transformed into straight-line segments. The approximation (1.3) ensures satisfaction of the deformation conditions

$$\varepsilon_{22} = \varepsilon_{33},\ \varepsilon_{23} = 0.$$

The condition $\varepsilon_{23} = 0$ also corresponds to the system of unit vectors $\mathbf{e}_i$ (i = 1, 2, 3) being orthogonal. We find the matrix of the transition from the basis $\mathbf{e}_i$ (i = 1, 2, 3) to the basis $\mathbf{e}_{i0}$ (i = 1, 2, 3),

$$\widehat{\mathbf{L}} = \begin{pmatrix} 1 + \partial w_1/\partial\varepsilon & \widehat{a}_2 & \widehat{a}_3 \\ \partial w_2/\partial\varepsilon & 1 + \widehat{b}_2 & \widehat{b}_3 \\ \partial w_3/\partial\varepsilon & -\widehat{b}_3 & 1 + \widehat{b}_2 \end{pmatrix},\ \widehat{\mathbf{\Omega}} = \begin{pmatrix} 0 & \widehat{a}_3 & -\widehat{a}_2 \\ 0 & \widehat{b}_3 & -(1 + \widehat{b}_2) \\ 0 & 1 + \widehat{b}_2 & \widehat{b}_3 \end{pmatrix},$$

with which at a given velocity $\mathbf{v}$ we project Eqs. (1.1) and (1.2) onto the axis of $\mathbf{e}_{i0}$ (i = 1, 2, 3):

$$m(\varepsilon)\frac{\partial^2\mathbf{w}}{\partial\tau^2} = \frac{\partial}{\sigma\varepsilon}(\widehat{\mathbf{L}}\mathbf{Q}) + \widehat{\mathbf{L}}\mathbf{R}; \tag{1.4}$$

$$\rho\frac{\partial\mathbf{l}^0}{\partial\tau} = \frac{\partial}{\partial\varepsilon}(\widehat{\mathbf{L}}\mathbf{M}) + \widehat{\mathbf{\Omega}}\mathbf{Q} + \widehat{\mathbf{L}}\boldsymbol{\mu}, \tag{1.5}$$

$$\mathbf{l}^0 = \{l_i^0 = J_i\omega_i,\ i = 1, 2, 3\},\ \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3).$$

For the components Q_i and M_i we find expressions in terms of the displacements of the points of the cross sections of the rod. Then from (1.4) and (1.5) we obtain rather cumbersome equations in the displacements. We make the necessary simplifications for the case under consideration. We henceforth consider the inertia of the rotational motions to be insignificant, i.e., we disregard the left sides of (1.5) and so assume that $\mu_a = 0$ and $\mu_{10} = 0$ (μ_{20} and μ_{30} are constants). We can ascertain that by necessity $M_1 = 0$. By assuming that the displacements in all three directions are the same in any section of the rod, we set $\hat{a}_2 = \hat{a}_3 = \hat{b}_2 = \hat{b}_3 = 0$ we go over to relations for the forces and momenta expressed in terms of the displacements of the points of the axial line of the rod. Suppose that P_H is the pressure of the fluid on the road at a depth H. When an external stream acts a curvature and additional distributed forces appear because of the variability of the cross section S and the strains. This is why in the simplifications of the equations we leave the term $-P_H(\partial S/\partial\varepsilon)(\partial^2\mathbf{w}/\partial\varepsilon^2)$, which corresponds to the internal forces of the rod. We ignore the terms of high orders of smallness. From (1.5) we find the relation between Q_i and M_i (i = 2, 3). As a result, we have three equations of motion of the system in the displacements of the points on the axial line of the rod. Assuming that the transverse motions have little effect on the longitudinal motions, we obtain the boundary-value problem of the process under study,

$$\frac{\partial^2 u_1}{\partial t^2} = \frac{\partial^2 u_1}{\partial s^2} - P_H^{(1)}\frac{\partial^2 u_1}{\partial s^2} + \frac{1}{n_0}\frac{\partial n_0}{\partial s}\frac{\partial u_1}{\partial s} + f_1,$$

$$\frac{\partial^2 u_2}{\partial t^2} = -\frac{\partial^4 u_2}{\partial s^4} - P_H^{(2)} \frac{\partial^2 u_2}{\partial s^2} + a_1 \frac{\partial}{\partial s}\left(\frac{\partial u_1}{\partial s}\frac{\partial u_2}{\partial s}\right) + a_2 \frac{\partial n_0}{\partial s}\frac{\partial u_1}{\partial s}\frac{\partial u_2}{\partial s} + a_3 \frac{\partial u_2}{\partial s} + f_2,$$
$$\frac{\partial^2 u_3}{\partial t^2} = -\frac{\partial^4 u_3}{\partial s^4} - P_H^{(3)} \frac{\partial^2 u_3}{\partial s^2} + b_1 \frac{\partial}{\partial s}\left(\frac{\partial u_1}{\partial s}\frac{\partial u_3}{\partial s}\right) + b_2 \frac{\partial u_0}{\partial s}\frac{\partial u_1}{\partial s}\frac{\partial u_3}{\partial s} + b_3 \frac{\partial u_3}{\partial s} + f_3 \tag{1.6}$$

with the boundary conditions

$$u_i(t, s)_{|s=0} = 0, \; i = 1, 2, 3, \; \frac{\partial^2 u_2}{\partial s^2}_{|s=0} = \frac{\partial^2 u_3}{\partial s^2}_{|s=0} = 0, \; \frac{\partial u_1}{\partial s}_{|s=1} =$$
$$= -c_1 \frac{\partial^2 u_1}{\partial t^2}_{|s=1} + c_2, \; \frac{\partial^2 u_2}{\partial s^2}_{|s=1} = 0, \; \frac{\partial^3 u_2}{\partial s^3}_{|s=1} = n \frac{\partial^2 u_2}{\partial t^2}_{|s=1}, \; \frac{\partial^2 u_3}{\partial s^2}_{|s=1} = \tag{1.7}$$
$$= n_1 (g\rho_f V_\Pi - q_\Pi), \; \frac{\partial^3 u_3}{\partial s^3}_{|s=1} = n_2 \frac{\partial^2 u_3}{\partial t^2}_{|s=1} - n_3 (g\rho_f V_\Pi - q_\Pi)$$

and initial conditions

$$u_i(t, s)_{|t=t_0} = k_i(s), \; \frac{\partial u_i(t, s)}{\partial t}_{|t=t_0} = g_i(s), \; i = 1, 2, 3. \tag{1.8}$$

Here the dimensionless time is $t = \tau \ell \sqrt{m/ES\delta}$ in the first equation while in the second equation it is $t = \tau \ell^2 \sqrt{m/EI_3\delta}$, in the third $t = \tau \ell^2 \sqrt{m/EI_2\delta}$; $P_H^{(1)} = P_H \frac{\partial n_0}{\partial s}\frac{1}{En_0}$; $P_H^{(2)} = P_H \frac{\partial n_0}{\partial s}\frac{S_0 l^2}{EI_3\delta}$; $P_H^{(3)} = P_H \frac{\partial n_0}{\partial s}\frac{S_0 l^2}{EI_2\delta}$; $a_1 = \frac{S_0 n_0 lh}{I_3}$; $a_2 = \frac{S_0 hl}{I_3}$; $a_3 = \frac{l^3 R_1}{EI_3\delta}$; $b_1 = \frac{S_0 n_0 hl}{I_2}$; $b_2 = \frac{S_0 lh}{I_2}$; $b_3 = \frac{l^3 R_1}{EI_2\delta}$; $f_1 = \frac{l^2 R_1}{ES_0 n_0 h\delta}$; $f_2 = \frac{l^4 R_2}{EI_3 h\delta}$; $f_3 = \frac{l^4 R_3}{EI_2 h\delta}$; $R_1 = \Omega - q_1 - F_1$; $R_2 = q_2 + F_2$; $R_3 = P_A - P_m + q_3 + \bar{F}_3$; $F_i = F_{ic} + F_{ip}$ $(i = 1, 2, 3)$; $c_1 = q_\Pi l \; (\delta g T^2 E S_B)^{-1}$; $c_2 = R_1 l (\delta E h S_B)^{-1}$; $n = q_\Pi l^3 (\delta g T^2 I_{3B})^{-1}$; $n_1 = l^2 \tilde{h} \xi_c (\delta E I_{2B} h)^{-1}$; $n_2 = q_\Pi l^3 (\delta g T^2 E I_{2B})^{-1}$; $n_3 = l^3 (\delta E h I_{2B})^{-1}$; $\delta = (1 - \nu)[(1 + \nu)(1 - 2\nu)]^{-1}$; T is the characteristic interval of time for Π. We assume that the problem (1.6)-(1.8) has a unique solution for given $g_i(s)$ and $k_i(s)$ (i = 1, 2, 3). The boundary conditions (1.7) were obtained as follows. In the leading section A the boundary conditions correspond to a swivel fastening of the rod with a propulsive device. In section B the body Π has a link to the rod along a plane perpendicular to the axis of the rod. We assume that Π has two symmetry planes, which pass through the unit vectors ($\mathbf{e}_1$, $\mathbf{e}_3$) and ($\mathbf{e}_1$, $\mathbf{e}_2$); body Π is assumed to be rigid. The center of mass C is on the line BCΠ of intersection of these planes and ξ_C = BC. The law of motion of point c is $\mathbf{x}_c(t) = (\mathbf{e}_1 - \mathbf{e}_{10})\xi_c + \mathbf{w}(l, t)$. The inertial force of the translational motion

$$J_\Pi = -\frac{q_\Pi}{g}\frac{d^2 \mathbf{x}_c}{dt^2} \equiv -\frac{q_\Pi}{g}\frac{\partial^2 \mathbf{w}}{\partial t^2}_{|s=l}.$$

The inertial forces of the rotational motions and the angles of rotation for Π are disregarded. We come to the conclusion that the torsional moment $\mathbf{M}_{1\Pi}$ and the bending moment $\mathbf{M}_{3\Pi}$ for Π can set equal to zero. Applying the d'Alembert principle to Π and carrying out manipulations, we obtain the conditions (1.7).

2. Conditions for the Technical Stability of the States of a Transported Rod in a Fluid. We consider the vector functional

$$V[u_1, u_2, u_3; t] = \{V_i[u_i, t], \; i = 1, 2, 3\}, \; V_1[u_1, t] =$$
$$= \int_0^1 ds \left[\left(\frac{\partial u_1}{\partial s}\right)^2 - (\tilde{v}_1 + \tilde{F}_1)\left(\frac{\partial u_1}{\partial s}\right)^2 + \left(\frac{\partial u_1}{\partial t}\right)^2\right], \; V_2[u_2, t] = \int_0^1 ds \left[\left(\frac{\partial^2 u_2}{\partial s^2}\right)^2 -\right.$$
$$\left. - (\tilde{v}_2 + \tilde{F}_2)\left(\frac{\partial u_2}{\partial s}\right)^2 + \left(\frac{\partial u_2}{\partial t}\right)^2\right], \; V_3[u_3, t] = \int_0^1 ds \left[\left(\frac{\partial^2 u_3}{\partial s^2}\right)^2 - (\tilde{v}_3 + \tilde{F}_3)\left(\frac{\partial u_3}{\partial s}\right)^2 + \right. \tag{2.1}$$
$$\left. + \left(\frac{\partial u_3}{\partial t}\right)^2\right], \; \tilde{v}_1 = \sup_s \left(\frac{mvl^2}{c^2 h}\right), \; \tilde{v}_2 = \sup_s \left(\frac{mvl^3}{\delta EI_3}\right), \; \tilde{v}_3 = \sup_s \left(\frac{mvl^3}{\delta EI_2}\right), \; \tilde{F}_k =$$
$$= \sup_s (P_H^{(k)} + f_k), \; k = 1, 2, 3, \; c^2 = ES_0 n_0(s)\delta$$

and the vector measure

$$\rho(\mathbf{u}) = \{\rho_i(u_i), i = 1, 2, 3\}, \ \rho_1(u_1) = \sup_s (u_1)^2 + \int_0^1 ds\left[\left(\frac{\partial u_1}{\partial s}\right)^2 + \left(\frac{\partial u_1}{\partial t}\right)^2\right],$$
$$\rho_j(u_j) = \sup_s (u_j)^2 + \sup_s \left(\frac{\partial u_j}{\partial s}\right)^2 + \int_0^1 ds\left[\left(\frac{\partial^2 u_j}{\partial s^2}\right)^2 + \left(\frac{\partial u_j}{\partial t}\right)^2\right], \ j = 2, 3. \quad (2.2)$$

For components of the vector function (2.1) we have the lower estimate

$$V_1[u_1, t] \geqslant \frac{1}{2}[1 - (\tilde{v}_1 + \tilde{F}_1)]\rho_1(u_1), \ V_i[u_i, t] \geqslant \frac{1}{3}[1 - (\tilde{v}_i + \tilde{F}_i)] \times \rho_i(u_i), \ i = 2, 3, \quad (2.3)$$

and the functionals $V_i|u_i, t|$ are positive-definite with respect to the measure $\rho(\mathbf{u})$ for $0 \le \tilde{v}_i + \tilde{F}_i < 1$, i = 1, 2, 3. The quantities $\mu_i = 1 - (\tilde{v}_i + \tilde{F}_i)$ (i = 1, 2, 3) are small parameters; $\mu_i \in (0, 1]$. We assign the finite intervals of time $I_1 = [t_0, L\bar{\mu}^{-1}]$, $\bar{\mu}^{-1} = \max\{\mu_1^{-1}, \mu_2^{-1}, \mu_3^{-1}\}$; L is a given, arbitrarily large constant, which characterizes the reliability of the system (L > 0).

Definition 1. The dynamic process described by the problem (1.6)-(1.8) is called the technical stability in a finite time interval I_1 with respect to a given measure $\rho(\mathbf{u})$ if along the perturbed solution $\mathbf{u}(t, s)$ of the problem (1.6)-(1.8) for the vector $V[\mathbf{u}, t]$ with positive-definite components $V_i[u_i, t]$ (i = 1, 2, 3) with respect to the corresponding components $\rho_i(u_i)$ (i = 1, 2, 3) of the measure $\rho(\mathbf{u})$ the conditions

$$V_i[u_i(t, s), t] \leqslant P_i(t), \ t \in I_1, \ i = 1, 2, 3,$$

are satisfied only at the initial instant

$$V_i[u_i(t_0, s), t_0] \leqslant b_i, \ t_0 \in I_1, \ i = 1, 2, 3, \quad (2.4)$$

where (2.4) is prescribed by the conditions (1.7), (1.8), and the bounded functions $P_i(t)$ (i = 1, 2, 3) defined in the domain I_1 satisfy the conditions

$$0 < P_i(t) \leqslant C_i, \ C_i = \text{const} > 0,$$
$$P_i(t_0) \geqslant b_i, \ b_i = \text{const} > 0, \ i = 1, 2, 3.$$

The functions $P_i(t)$ (i = 1, 2, 3), the constants C_i, b_i (i = 1, 2, 3), and I_1 are given beforehand.

Definition 2. The process (1.6)-(1.8) is called the technical stability in a finite time interval I, when the conditions of Definition 1 hold for any $K \le +\infty$. If in this case

$$\lim_{t \to +\infty} V_i[u_i(t, s), t] = 0, \ i = 1, 2, 3,$$

the process (1.6)-(1.8) is said to be technically asymptotically stable.

Definition 3. The process (1.6)-(1.8) is said to be technically unstable in a finite or infinite time interval for given constants b_i and the functions $P_i(t)$, when conditions (2.4) are satisfied a value $t_1 \in I_1$ or $t_1 \in I$ $(t_1 > t_0)$ is obtained for the solution $\mathbf{u}(t, s)$, this value being such that at least one of the inequalities

$$V_i[u_i(t_1, s), t_1] > C_i, \ C_i = \text{const} > 0, \ i = 1, 2, 3$$

is satisfied.

It follows from Definitions 1-3 that the conditions for technical stability differ substantially from the Lyapunov stability properties in that not only is the system considered in any given finite time interval but also the constraints on the initial states of the process do not depend on the conditions of the prescribed majorization of the subsequent states of the process during the given time interval. The fact that the complete derivative of the Lyapunov functional on the basis of the boundary-value problem need not necessarily be negative definite, in contrast to the case of Lyapunov stability, expands the range of values to the parameters of the process under study.

By virtue of (1.6)-(1.8) the total derivative of $V[u_1, u_2, u_3; t]$ with respect to t has the form

$$\frac{dV_1[u_1, t]}{dt} = 2\left\{\frac{\partial u_1}{\partial t}(t, 1)\left[c_2 - c_1\frac{\partial^2 u_1}{\partial t^2}(t, 1)\right] - \int_0^1 ds\left[(\tilde{v} + \tilde{F}_1)\frac{\partial u_1}{\partial s}\frac{\partial^2 u_1}{\partial t\,\partial s} + \left(P_H^{(1)}\frac{\partial^2 u_1}{\partial s^2} - n_0^{-1}\frac{\partial u_1}{\partial s}\frac{\partial n_0}{\partial s} - f_1\right)\frac{\partial u_1}{\partial t}\right]\right\}, \quad \frac{dV_2[u_2, t]}{dt} = -2n\frac{\partial u_2}{\partial t}(t, 1)\frac{\partial^2 u_2}{\partial t\,\partial s}(t, 1) + 2 \times$$

$$\times \int_0^1 ds\left\{\frac{\partial u_2}{\partial t}\left[a_1\frac{\partial}{\partial s}\left(\frac{\partial u_1}{\partial s}\frac{\partial u_2}{\partial s}\right) - P_H^{(2)}\frac{\partial^2 u_2}{\partial s^2} + a_2\frac{\partial n_0}{\partial s}\frac{\partial u_1}{\partial s}\frac{\partial u_2}{\partial s} + a_2\frac{\partial u_2}{\partial t} + f_2\right] - (\tilde{v}_2 + \tilde{F}_2)\frac{\partial u_2}{\partial s}\frac{\partial^2 u_2}{\partial t\,\partial s}\right\}, \tag{2.5}$$

$$\frac{dV_3[u_3, t]}{dt} = 2\left[\left(n_1\frac{\partial^2 u_3}{\partial t\,\partial s}(t, 1) + n_3\frac{\partial u_3}{\partial t}(t, 1)\right)(g\rho_f V_\Pi - q_\Pi) - n_2\frac{\partial u_3}{\partial t}(t, 1) \times \frac{\partial^2 u_3}{\partial t^2}(t, 1) + 2\int_0^1 ds\left\{\frac{\partial u_3}{\partial t}\left[b_1\frac{\partial}{\partial s}\left(\frac{\partial u_1}{\partial s}\frac{\partial u_3}{\partial s}\right) + P_H^{(3)}\frac{\partial^2 u_3}{\partial s^2} + b_2\frac{\partial n_0}{\partial s}\frac{\partial u_1}{\partial s}\frac{\partial u_3}{\partial s} + b_3\frac{\partial u_3}{\partial s} + f_3\right] - (\tilde{v}_3 + \tilde{F}_3)\frac{\partial u_3}{\partial s}\frac{\partial^2 u_3}{\partial t\,\partial s}\right\}.$$

We denote Eq. (2.5) on the right, respectively, by $M_1(t, \lambda_1)$, $M_2(t, \lambda_2)$, $M_3(t, \lambda_3)$, where the parameters $\lambda_1 = (c_1, c_2, \tilde{v}_1, \tilde{F}_1, P_H^{(1)}, f_1)$, $\lambda_2 = (n, a_1, a_2, a_3, \tilde{v}_2, \tilde{F}_2, P_H^{(2)}, f_2)$, $\lambda_3 = (n_1, n_2, n_3, b_1, b_2, b_3, \tilde{v}_3, \tilde{F}_3, P_H^{(3)}, f_3)$ characterize the system (1.6)-(1.8). We consider the function

$$\overline{\Phi_1(t, \lambda_1)} = M_1(t, \lambda_1) - \frac{\mu_1}{2(\mu_1 + t)^2}\rho_1(u_1(t, s)), \quad \overline{\Phi_i(t, \lambda_i)} = M_i(t, \lambda_i) - \frac{\mu_i}{3(\mu_i + t)^2}\rho_i(u_i(t, s)), \quad i = 2, 3.$$

For the prescribed nonnegative functions $\Phi_i(t)$ ($i = 1, 2, 3$) that are integrable over t require satisfaction of the conditions

$$|\overline{\Phi_i(t, \lambda_i)}| \leqslant \Phi_i(t), \quad i = 1, 2, 3.$$

We can choose $\Phi_i(t) = e^{\alpha_i(t)}$ [$\alpha_i(t)$ are continuous functions, $t \in I_1 \subset I$], in particular we can set $\alpha_i(t) = 1/(\mu_i + t)$ or $\alpha_i(t) = -2/(\mu_i + t)$. We introduce the notation $\sigma_i(t) = \int_{t_0}^{t}\Phi_i(\tau)\,d\tau$ ($i = 1, 2, 3$). We consider the functions $z_i(t) = V_i[u_i(t, s), t] - \sigma_i(t)$ ($i = 1, 2, 3$) along the solutions of the problem (1.6)-(1.8). The estimates for dV_i/dt ($i = 1, 2, 3$) along the solutions of this problem lead to the system of inequalities [9-11]

$$\frac{dz_i(t)}{dt} \leqslant \frac{1}{(\mu_i + t)^2}[z_i(t) + \sigma_i(t)], \quad i = 1, 2, 3. \tag{2.6}$$

Equation (2.6) infers Cauchy comparisons of the form

$$\frac{dy_i}{dt} = \frac{1}{(\mu_i + t)^2}[y_i + \sigma_i(t)], \quad t \in I_1, \quad i = 1, 2, 3; \tag{2.7}$$

$$y_i(t_0) = y_i^0 \geqslant V_i[u_i(t_0, s), t_0], \quad t_0 \in I_1, \quad i = 1, 2, 3. \tag{2.8}$$

The functions appearing in $V_i[u_i(t_0, s), t_0]$ were determined by conditions (1.7), (1.8) of problem (1.6)-(1.8). In region I_1 problem (2.7), (2.8) has a continuous solution

$$y_i(t) = \exp[-1/(\mu_i + t)]\int_{t_0}^{t}\exp[1/(\mu_i + \tau)]\,\Phi_i(\tau)\,d\tau + y_i^0\exp[1/(\mu_i + t)] \times \exp[-1/(\mu_i + t)] - \sigma_i(t), \quad i = 1, 2, 3.$$

From the theorem of differential inequalities [11] we find

$$z_i(t) \leqslant y_i(t),\ i=1,\ 2,\ 3,\ t \in I_1.$$

Along the solution of problem (1.6)-(1.8) we have the estimates

$$V_i[u_i(t, s), t] \leqslant P_i(t),\ P_i(t) \equiv \exp[-1/(\mu_i+t)] \int_{t_0}^{t} \exp[1/(\mu_i+\tau)] \times \\ \times \Phi_i(\tau)\,d\tau + y_i^0 \exp[1/(\mu_i+t_0)] \exp[-1/(\mu_i+t)],\ t_0,\ t \in I_1,\ i=1, 2, 3. \tag{2.9}$$

Suppose that the functions $\Phi_i(t)$ satisfy the conditions

$$\int_{t_0}^{t} \Phi_i(\tau) \exp[1/(\mu_i+\tau)]\,d\tau \leqslant M_i(\mu_i + L\bar{\mu}^{-1})^2 \{\exp[1/(\mu_i+t_0)] - \\ - \exp[1/(\mu_i+L\bar{\mu}^{-1})]\},\ t_0,\ t \in I_1,\ i=1, 2, 3,$$

where the prescribed constants M_i satisfy the condition $|M_i(t, \lambda_i)| \leq M_i$, i = 1, 2, 3. Hence we obtain the system of estimates

$$P_i(t) \leqslant C_i \equiv M_i(\mu_i + L\bar{\mu}^{-1})^2 + y_i^0 \exp[1/(\mu_i+t_0)],\ t_0,\ t \in I_1,\ i=1, 2, 3.$$

By virtue of (2.8), (2.9), therefore, the process (1.6)-(1.8) is technically stable in region I_1 with respect to the measure $\rho(\mathbf{u})$. When the boundary-value problem (1.6)-(1.8) is determined in any interval $I_1 \subseteq I$ and estimates (2.8), (2.9) are valid in each region $I_1 \subseteq I$, the process is technically stable in an infinite time interval. In particular, we have this property of the process when the integrals $\int_{t_0}^{t} \exp[1/(\mu_i+\tau)]\Phi_i(\tau)\,d\tau$ $(i=1,\ 2,\ 3)$ are continuous functions in any interval $I_1 \subseteq I$ and increase in each $I_1 \subseteq I$ no faster than the corresponding functions

$$N_i \int_{t_0}^{t} (\mu_i+\tau)^{-2} \exp[1/(\mu_i+\tau)]\,d\tau,\ N_i = \text{const} > 0,\ N_i \leqslant M_i,\ i=1,2,3.$$

Satisfaction of the conditions

$$\exp[-1/(\mu_i+t)] \geqslant \int_{t_0}^{t} \exp[1/(\mu_i+\tau)]\Phi_i(\tau)\,d\tau,\ I_1 \subseteq I,\ i=1, 2, 3,$$

or

$$\exp[1/(\mu_i+t)] \geqslant \int_{t_0}^{t} \exp[1/(\mu_i+\tau)]\Phi_i(\tau)\,d\tau,\ I_1 \subseteq I,\ i=1, 2, 3,$$

ensures the technical stability of the initial process (1.6)-(1.8) in an infinite time interval if in $M_i(\mu_i + L\bar{\mu}^{-1})^2 \geq 1$, or when

$$P_i(t) \leqslant y_i^0 \exp[1/(\mu_i+t_0)] + 1 \leqslant C_i,\ t_0,\ t \in I,\ i=1, 2, 3.$$

If in addition to the technical stability of the process (1.6)-(1.8) the conditions

$$P_i(t) \to 0,\ t \to +\infty,\ i=1, 2, 3, \tag{2.10}$$

are satisfied in region I, then the initial process is technically asymptotically stable with respect to the measure $\rho(\mathbf{u})$. In particular, (2.10) obtains, if

$$\exp[-1/(\mu_i+t)] \int_{t_0}^{t} \exp[1/(\mu_i+\tau)]\Phi_i(\tau)\,d\tau \to -y_i^0 \exp[1/(\mu_i+t_0)], \\ t \to +\infty,\ i=1, 2, 3.$$

The indicated conditions for the technical stability of the system are violated if the velocity of the rod and the external forces acting on the rod satisfy the system of inequalities

$$\tilde{v}_i + \tilde{F}_i \geqslant 1, \ i = 1, 2, 3, \tag{2.11}$$

since in this case the positive definiteness condition (2.3) for the functional (2.1) is not satisfied. But this is insufficient for the system to be unstable. It is technically unstable in I_1 or in I if in these regions the respective majorants $P_i(t)$ in (2.9) satisfy the conditions

$$P_i(t) \to +\infty, \ i = 1, 2, 3. \tag{2.12}$$

In particular, (2.12) obtains for $t_0 = 0$ and arbitrary $t \geq 0$ when $\mu_i \to 0$ (i = 1, 2, 3), as follows from determination of μ_i(i = 1, 2, 3), this is possible when the velocity of the rod in the fluid tends toward a critical value v_{cr}, which also applies in similar fashion to the external forces acting on the rod, since these quantities increase simultaneously. In the given case v_{cr} in an ideal fluid is determined by the inequalities (2.11):

$$v_{cr} = \Big[3ES_0n_0hI_2I_3\delta - I_2I_3\left(R_1l^2 + S_0h\delta P_H\frac{\partial n_0}{\partial s}\right) - \\ - S_0n_0I_2\left(R_2l^4 + P_HS_0l^2h\frac{\partial n_0}{\partial s}\right) - S_0n_0I_3\left(R_3l^4 + P_HS_0l^2h\frac{\partial n_0}{\partial s}\right)\Big] \times \\ \times [ml^2(I_2I_3 + lS_0n_0hI_2 + lS_0n_0hI_3]^{-1}. \tag{2.13}$$

For example, v_{cr} = 38 km/h for ℓ = 2 km, v_{cr} = 53 km/h for ℓ = 1 km, and v_{cr} = 75 km/h for ℓ = 0.5 km for the corresponding other parameters in (2.13).

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